Notes on the Cœlostat and Siderostat. By H. C. Plummer, M.A.

1. At the present time the desirability of using telescopes of exceedingly great focal length is urgently felt. There is no insuperable difficulty in making reflectors or lenses with focal lengths of a hundred feet or more, but it is practically impossible to mount instruments of this size in such a way that they can be directed to any part of the sky and made to follow the diurnal motion of the stars. They must be placed in stationary positions, and the simplest construction is possible when the telescope is horizontal. In order that any part of the sky may be observed in the telescope, it is necessary to place a plane mirror in front of the reflector or object-glass. The mirror may be mounted in such a way that as a particular star describes its diurnal path, its image remains fixed in the field of the telescope. If this is the case the motion of the mirror must be of a simple and definite geometrical type (cf. *Monthly Notices*, vol. lxi. p. 459) which may be called siderostatic, although the motion of the siderostat may be produced by many different forms of mechanism. The use of a siderostat, notwithstanding the advantages which it offers, is not free from objection. In spite of the ingenuity which has been devoted to several forms of construction it has been generally felt that the mechanical design of the instrument leaves much to be desired. The largest and most elaborate siderostat yet made was that shown at the Paris Exhibition, 1900; but unfortunately no information seems to have been published on which a reliable judgment of its performance can be based. Dr. Johnstone Stoney (*Monthly Notices*, vol. lvi. p. 452) has suggested a way in which the chief mechanical defect of the older forms may be avoided, but I am not aware that the suggestion has been embodied in an instrument of sufficient size to test its practical value.

2. In addition to the admitted defects of the siderostat itself, a further difficulty arises from the variable rotation of the field. For photographic purposes the rotation may be compensated by one of several devices which have been suggested for turning the plate. But however complete the mechanical arrangements there will be the same ultimate need for hand-control as is found with the equatorial, owing to inequalities of driving and the effect of refraction. In this case it is necessary to watch not one, but two stars in order to control the orientation of the field. It will probably be found that two eyepieces and two observers are required. Hence the advantage of using a cœlostat—which is that limiting form of siderostat which gives a field of constant orientation—is very evident. Moreover the uniform rotation of the mirror about a fixed axis requires the simplest possible mechanism. But on the other hand a fixed telescope in conjunction with a cœlostat commands only one definite declination in the sky, and other declinations must be reached by moving

the telescope either actually or virtually, the latter being the case when the rays reflected by the cœlostat are deflected to the fixed telescope by a subsidiary plane mirror. The Snow telescope of the Yerkes Observatory is an example of the use of a second mirror in this way. The conditions on which the efficient working of the combination depends deserve some consideration, for no investigation of them seems to have been published. Before examining this question, however, it is intended to discuss the possibilities presented by a single mirror which is mounted in a simple way—namely, such as to allow independent rotations about two orthogonal axes.

3. Let us suppose in the first place that the mirror is equatorially mounted, the normal to the mirror corresponding to the axis of the telescope; so that the normal can be adjusted to any desired N.P.D. and driven in right ascension by clockwork. The instrument becomes an ordinary cœlostat when the N.P.D. is 90°, and the velocity in R.A., N, is half the actual

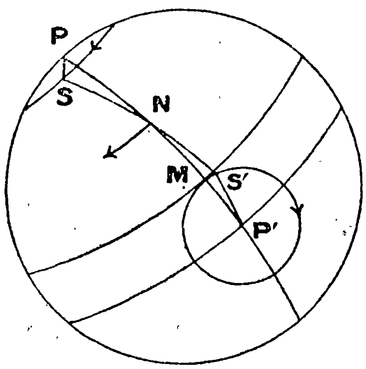

FIG. 1.

angular velocity of the sky, n. By adjusting the declination suitably any desired object can be reflected into the field of the telescope, and the combination can be used for visual observations or for photography when a practically instantaneous exposure is sufficient.

When the normal is inclined to the polar axis at a constant angle ν, and the mirror is driven at a uniform angular velocity N, it is easy to find a simple geometrical representation of the motion of the field. Let it be supposed for the moment that the opposite velocity, $-$N, is impressed on the mirror and the stars. The mirror is thus brought to rest, while a star S (fig. 1) describes its circle round the pole P with the angular velocity $n-\text{N}$. Let the ray from P be reflected in the direction P′ and the ray from S in the direction S′. The spherical triangles SNP, S′NP′ are altogether equal and consequently S′ describes round P′ a circle with the same radius and the same velocity $n-\text{N}$ as S describes round P.

Now let the angular velocity N be restored to the stars and

the mirror. Then P′ rotates uniformly round P at a distance 2ν with the angular velocity N, carrying with it a picture of the sky which rotates uniformly with respect to PP′ with the velocity $n-N$.

If $PS=P'S'=\Delta$, the circle whose centre is P′ and radius Δ touches the circle whose centre is P and radius $2\nu-\Delta$. Let M be the point of contact. Owing to the rotation of M round P′, the linear velocity of M on the sphere has a component proportional to $(n-N)\sin\Delta$ in one direction, and owing to the rotation of PP′ about P, it has a component proportional to $N\sin(2\nu-\Delta)$ in the opposite direction. Hence if

$$(n-N)\sin\Delta = N\sin(2\nu-\Delta)$$

or

$$\frac{N}{n}=\frac{\sin\Delta}{2\sin\nu\cos(\nu-\Delta)} \quad \ldots \quad \ldots \quad (1)$$

the point M is instantaneously at rest. It follows that the motion of the field may be represented as due to the rolling of the spherical cap whose centre is P′ and radius Δ on the rim of the spherical cap whose centre is P and radius $2\nu-\Delta$. The reflected ray from every star therefore traces a spherical roulette. The motion can also be ascribed to the rolling of one right circular cone on another whose axis is the polar axis.

4. In the case of the properly adjusted cœlostat $\nu = 90°$ and $n = 2N$, so that equation (1) is satisfied identically. The moving circle rolls on itself and the field is permanently stationary, so that a telescope pointed at the mirror will always see the same stars. If the rate of driving is $\frac{1}{2}n$ as for the cœlostat, but the N.P.D. of the normal to the mirror, ν, is not 90°, the rolling and fixed circles are equal but not coincident, and the radius of each is ν. If (ρ, θ) are the polar coordinates of S′, the sides of the triangle S′PP′ are ρ, 2ν, and Δ; and if the angle at P′ is χ, the angle at P is $\chi-\theta$. Hence by eliminating the angle at S′ from two of Delambre's analogies which involve the difference of the angles at P and P′, the polar equation of the locus of S′ on the sphere is easily found to be

$$\sin^2\rho+\sin^2\Delta-2\sin\Delta\sin\rho\cos\theta = \cot^2\nu\,(\cos\rho-\cos\Delta)^2$$

The orthogonal projection of this curve on the equatorial plane can be found in polar coordinates by putting

$$r\cos\phi = \sin\rho\cos\theta-\sin\Delta,\quad r\sin\phi=\sin\rho\sin\theta$$

The result of making this substitution is to obtain the ordinary equation of a *limaçon* referred to its pole, which is clearly the projection of the point obtained when $\nu=\frac{1}{2}\pi$, *i.e.* with a cœlostat in perfect adjustment. This theorem is due to Professor Turner (*Monthly Notices*, lvi. p. 419).

5. Since in the general case the motion of the sky as viewed

in the rotating mirror is that which arises from the rolling of one circle on another, it appears that the mirror introduces a complexity which is absent in the simple rotation of the sky as viewed directly. But the fact that a star on the instrumental meridian can be reflected in a chosen direction and by a suitable rate of driving kept stationary even for an instant suggests that it is not without interest to examine the state of things when the star is followed to some distance from the meridian. By a proper adjustment of the rate of driving, provided the clock is capable of a considerable range in this respect, the reflexion of any star can be maintained on the meridian, and its motion in this plane may be compensated by moving the plate-holder. The effect is comparable with that which would be caused by an exaggerated refraction. There is also a rotation of the field, but this would equally be the case if a siderostat were used. Consequently if the motion in declination of the reflected ray could be shown not to exceed a manageable amount, the method would appear to be practicable, and the extremely simple mounting of the mirror gives the question some practical importance.

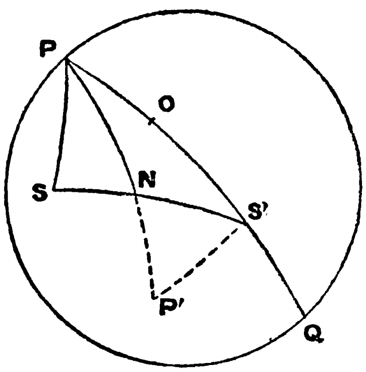

FIG. 2.

6. Let PQ (fig. 2) represent the instrumental meridian, N the direction of the normal to the mirror, S′ the reflexion of the star S, and P′ the reflexion of the pole P, so that $PP' = 2\nu$ and $P'S' = PS = \Delta$. Let $PS' = \rho$, $QPP' = H$, and $QPS = h$; hence $PP'S' = SPP' = h - H$. The relations which are required are arrived at by considering the triangle S′PP′. The first is the relation between H and h necessary in order that S′ may remain on the fixed meridian. This is

$$\cot \Delta \sin 2\nu = \cos 2\nu \cos (h-H) + \sin (h-H) \cot H$$

which can be put in a form to deduce h from H thus:

$$\cot \Delta \sin 2\nu \sin H = \sin h (\cos 2\nu \sin^2 H + \cos^2 H) - \cos h \sin^2 \nu \sin 2H$$

$$\left.\begin{aligned} &= \sin (h-K) \,.\, (1 - \sin^2 H \sin^2 2\nu)^{\frac{1}{2}} \\ \text{where} \quad \sin K &= \sin 2H \sin^2 \nu \,.\, (1 - \sin^2 H \sin^2 2\nu)^{-\frac{1}{2}} \end{aligned}\right\} \ldots \quad (2)$$

The corresponding clock-rate is obtained by differentiating the first equation between h and H. This gives

$$\frac{N}{n}=\frac{dH}{dh}=\frac{2\sin^2\nu\sin^2 H\sin(h-H)+\sin H\cos h}{2\sin^2\nu\sin^2 H\sin(h-H)+\cos H\sin h}$$

or

$$(n+N)/(n-N)=4\sin^2\nu\sin^2 H+\sin(h+H)/\sin(h-H)\ \ldots\ (3)$$

The displacement of the point S′ on the meridian, reckoned from the position corresponding to S on the meridian, is $\rho+\Delta-2\nu$. Now one of Napier's analogies gives

$$\tan\tfrac{1}{2}(\rho+\Delta)=\cos\tfrac{1}{2}(h-2H)\sec\tfrac{1}{2}h\tan\nu\qquad\ldots\ (4)$$

which leads to

$$\sin\tfrac{1}{2}(\rho+\Delta-2\nu)/\sin\tfrac{1}{2}(\rho+\Delta+2\nu)=\tan\tfrac{1}{2}H\tan\tfrac{1}{2}(h-H)\ \ldots\ (5)$$

This shows that the displacement may be very small. The angle through which the field has turned since S passed the meridian is QS′P′, which may be denoted by θ. Then clearly

$$\sin\theta=\sin 2\nu\sin H/\sin\Delta\qquad\ldots\qquad\ldots\ (6)$$

and

$$\cos\theta\,.\,\frac{d\theta}{dh}=\frac{\sin 2\nu\cos H}{\sin\Delta}\,.\,\frac{N}{n}\ \ldots\qquad\ldots\ (7)$$

The formulæ of this paragraph are all exact and suffice to determine with accuracy the relations between the clock-rate and the diurnal motion of the stars, the displacement of the reflected image in the plane of the instrumental meridian and the variable rotation of the field.

7. For our present purpose, however, which is simply to inquire whether the instrument is practicable for comparatively short exposures near the meridian, approximate formulæ are sufficient. The form of the equations shows that the neglect of powers of h higher than the first will not materially affect the results to be expected. Thus from (2)

$$K=2H\sin^2\nu$$
$$h-K=H\cot\Delta\sin 2\nu$$

whence

$$H/h=\sin\Delta/2\sin\nu\cos(\nu-\Delta)\ \ldots\qquad\ldots\ (8)$$

By (3) this is also the value of N/n, a result already found in equation (1). The displacement in the plane of the meridian is given by (5) in the form

$$\rho+\Delta-2\nu=\tfrac{1}{2}H(h-H)\sin\tfrac{1}{2}(\rho+\Delta+2\nu)$$

N N

which becomes, when ρ is given its initial value on the right-hand side,

$$\rho + \Delta - 2\nu = h^2 \sin\Delta \sin(2\nu - \Delta)/4 \tan\nu \cos^2(\nu - \Delta) \quad \dots (9)$$

The rate of rotation of the field, as given by (7), is

$$\frac{d\theta}{dh} = \frac{\cos\nu}{\cos(\nu - \Delta)} \quad \dots \quad \dots \quad \dots \quad \dots \quad (10)$$

This rotation is relative to the fixed meridian PQ, while that found in § 3 is relative to the moving meridian PN, and has a different value, namely, $n - N$.

8. Some numerical results can now be given. In the case chosen for the purpose of illustration the latitude is 50° and the telescope is placed in a horizontal position in the local meridian. Hence $\rho = 130°$ initially. The column headed R_1 gives the driving-rate of the mirror in R.A. compared with the corresponding clock-rate of an equatorial. The column headed R_2 contains the initial rotation of the field, expressed as a fraction of one complete rotation in a day. Under the heading D is given the displacement of the image of the guiding star in the first minute after passing the meridian.

Δ	ν	R_1	R_2	D
10°	70°	0·1848	+0·6840	+0″·1901
20	75	·3087	·4512	·2095
30	80	·3949	·2701	·1605
40	85	·4563	+0·1233	+0·0846
50	90	·5000	0·0000	0·0000
60	95	·5306	−0·1064	−0·0849
70	100	·5509	·2005	·1662
80	105	·5625	·2856	·2416
90	110	·5662	·3640	·3100
100	115	·5625	·4375	·3702
110	120	·5509	·5077	·4207
120	125	0·5306	−0·5758	−0·4596

The case of $\Delta = 50°$, in which the N.P.D. of the star is equal to the latitude of the place, satisfies the cœlostat condition. This case marks a change of sign in R_2 and D. The meaning of a positive sign attached to R_2 is that the rotation of the field is clockwise when viewed towards the mirror, the meaning of a negative sign that the rotation is anti-clockwise. Similarly a positive sign attached to D indicates that the displacement after the guiding star has passed the meridian is in the direction of the S. pole, a negative sign that it is towards the N. pole. Within a small distance of the meridian, the total shift of the guiding star

is obtained by multiplying the numbers in the last column by the square of the number of minutes in the hour-angle. The inferences to be drawn from the table are clearly uncertain in the absence of actual experiment. But it would appear at least feasible, with an instrument of the kind considered, to obtain photographs with moderate exposures (up to 20 minutes say) of a belt of the sky extending perhaps 30° south of the zenith.

9. Very steady driving may be expected from a mirror with simple equatorial mounting, and it would probably be best to adjust the clock-rate at the beginning of the exposure of the plate and to apply the adjustment afterwards necessary to the plate-carrier by means of two rectangular slides. By moving the normal to the mirror in N.P.D. the siderostat condition might be partially or completely satisfied, and it would be necessary to apply merely a rotation to the plate. This would complicate the mounting of the mirror, and would probably interfere with the steadiness of the driving. Yet it is a matter of interest to examine the nature of the motions which must be communicated to a mirror capable of rotating about two orthogonal axes in order that it may satisfy the siderostat condition. This problem differs from the one previously considered merely in the fact that ρ is constant and ν variable, instead of the converse. The triangle PS′P′ (fig. 2) has for its angles H, $\pi-\theta$ and $h-\mathrm{H}$, and for its sides Δ, 2ν and ρ. Hence three of Napier's analogies give immediately

$$\tan\tfrac{1}{2}\theta = \frac{\cos\tfrac{1}{2}(\rho+\Delta)}{\cos\tfrac{1}{2}(\rho-\Delta)}\tan\tfrac{1}{2}h \qquad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$\tan\nu\cos\tfrac{1}{2}(h-2\mathrm{H}) = \cos\tfrac{1}{2}h\tan\tfrac{1}{2}(\rho+\Delta)$$

$$\tan\nu\sin\tfrac{1}{2}(h-2\mathrm{H}) = \sin\tfrac{1}{2}h\tan\tfrac{1}{2}(\rho-\Delta)$$

The first of these gives the law of rotation of the field, discussed by Cornu. The other two equations lead to

$$\left.\begin{aligned}\tan\nu\sin\mathrm{H} &= \frac{\sin\Delta\sin h}{\cos\rho+\cos\Delta}\\ \tan\nu\cos\mathrm{H} &= \frac{\sin\rho+\sin\Delta\cos h}{\cos\rho+\cos\Delta}\end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (12)$$

or

$$\left.\begin{aligned}\tan\nu &= \frac{(\sin^2\rho+\sin^2\Delta+2\sin\rho\sin\Delta\cos h)^{\frac{1}{2}}}{\cos\rho+\cos\Delta}\\ \tan\mathrm{H} &= \frac{\sin\Delta\sin h}{\sin\rho+\sin\Delta\cos h}\end{aligned}\right\} \quad \ldots \quad (13)$$

which express the laws of motion about the two axes.

10. From these may be deduced the character of the motion about two axes, of which one is fixed in the instrumental meridian plane and inclined to the polar axis at an angle α. Let O represent (fig. 2) the direction of the fixed axis, and let

N N 2

ON $= \nu'$ and QON $=$ H′. The form of equations (12), which give the "standard coordinates" of the direction of the normal to the mirror, suggests a transformation in rectangular coordinates. In the polar system

$$x = \sin\Delta\sin h, \quad y = \sin\rho + \sin\Delta\cos h, \quad z = \cos\rho + \cos\Delta$$

If the axes are turned through an angle a about the axis of x,

$$x' = x, \quad y' = y\cos a - z\sin a, \quad z' = y\sin a + z\cos a$$

In the new system, as in the old,

$$\tan\nu'\sin H' = x'/z', \quad \tan\nu'\cos H' = y'/z'$$

Hence the motion of the mirror, with respect to the new axes, is given by

$$\left.\begin{aligned} \tan\nu'\sin H' &= \frac{\sin\Delta\sin h}{\cos(\rho-a)+\cos a\cos\Delta+\sin a\sin\Delta\cos h} \\ \tan\nu'\cos H' &= \frac{\sin(\rho-a)-\sin a\cos\Delta+\cos a\sin\Delta\cos h}{\cos(\rho-a)+\cos a\cos\Delta+\sin a\sin\Delta\cos h} \end{aligned}\right\} \quad (14)$$

11. From these results a number of particular cases can be deduced by assigning different values to a. If, for example, the instrumental is coincident with the local meridian, and a is the co-latitude, the equations (14) characterise the motion when the mirror has an altazimuth mounting. But no cases beyond those already known appear to present simple types of motion which can be derived from clockwork without complicated mechanism. Hence, unless such motions can be obtained by electric motor-driving under automatic control, the problem of the siderostat is apparently not facilitated by resolving the motion of the mirror about two orthogonal axes. This is true even if a is allowed to be a function of Δ, *i.e.* if the main axis, instead of being fixed, is adjusted according to the declination required—an arrangement only practicable with a small mirror. One case of some interest is that in which the fixed axis is directed towards the centre of curvature of the sphero-conic traced by the normal to the mirror, corresponding to the point where the meridian is crossed. The expressions occurring in equations (14) may be expanded in powers of $T = \tan\frac{1}{2}h$, thus

$$\tan\nu'\sin H' = A_1T + A_3T^3 + \ldots, \quad \tan\nu'\cos H' = B_0 + B_2T^2 + \ldots$$

so that

$$\tan^2\nu' = B_0^2 + (A_1^2 + 2B_0B_2)T^2 + \ldots$$

The coefficient of T^2 involves a, and can be made to vanish by a suitable choice of this angle. The result of a little reduction is to give the value

$$a = \tfrac{1}{2}\rho - \tan^{-1}(\tan\tfrac{1}{2}\Delta\tan^2\tfrac{1}{2}\rho) \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

It follows that when the fixed axis has the position indicated by this equation, the variation of ν' depends on the fourth power of the time, and is at first very slow. When the example of § 8, in which $\rho = 130°$, is examined, equation (15) gives the following numerical results:

Δ	α	Δ	α	Δ	α
$10°$	$43° \; 5'$	$50°$	$0° \; 0'$	$90°$	$12° \; 44'$ S.P.
20	25 58	60	4 22 S.P.	100	14 40 „
30	14 4	70	7 45 „	110	16 21 „
40	5 51	80	10 28 „	120	17 51 „

12. The advantage of the form of mounting which has been examined lies in the stability which it renders possible, a feature in regard to which the ordinary form of siderostat is liable to grave suspicion. It is not impossible that an entirely satisfactory method may be available for producing the motions defined by equations (13). It is easy to devise a linkage which contains the solution of the problem, and though link-motions are not free from objection, any method which is based on a new principle, as this appears to be, possesses a certain interest. Let O and C (fig. 3)

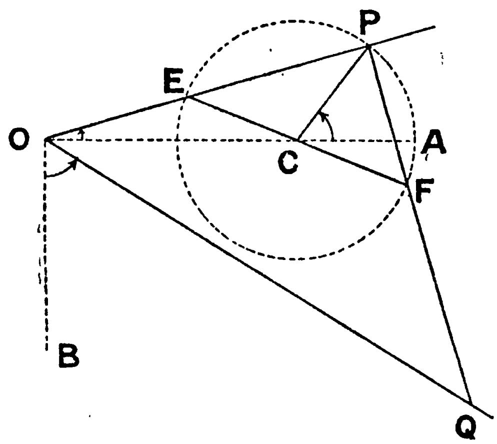

FIG. 3.

be two points fixed at a distance apart equal to $\sin \rho$. Let CP, CE and CF be each equal to $\sin \Delta$ on the same scale and PQ to $\cos \rho + \cos \Delta$. The lines OP, PQ, and OQ represent slotted bars, so that the points P and E are free to move along OP, F along PQ, and Q along OQ. Then if CP rotates uniformly in twenty-four hours so that the angle ACP is h, A lying on OC produced, we have

$$\tan \text{AOP} = \sin \Delta \sin h / (\sin \rho + \sin \Delta \cos h)$$

$$\tan \text{OQP} = \text{OP}/\text{PQ}$$

$$= (\sin^2 \rho + \sin^2 \Delta + 2 \sin \rho \sin \Delta \cos h)^{\frac{1}{2}} / (\cos \rho + \cos \Delta)$$

for OPQ is a right angle. Hence AOP is the angle H and OQP is the angle ν. If OB is perpendicular to OA, the angle BOQ is

equal to $\nu + H$. It follows that if the point O is on the polar axis and the plane of the linkage is perpendicular to this axis, a wheel of which OP is a radius rigidly connected with the mirror will communicate the required motion in R.A., while a second wheel of which OQ is a radius, if suitably geared round the frame to a wheel on the moving axis, will communicate the required motion in N.P.D. This constitutes a solution of the siderostat problem. The adjustment for different declinations involves the ratios of OC, CP and PQ. Probably the most convenient method would be to keep CP constant and to vary OC and PQ.

13. The use of a cœlostat in conjunction with a second mirror, which deflects the rays from the cœlostat into a permanently fixed telescope, has been mentioned in § 2, and the possibilities of this arrangement may now be considered. The position of the telescope is supposed to be horizontal, and can be specified by the

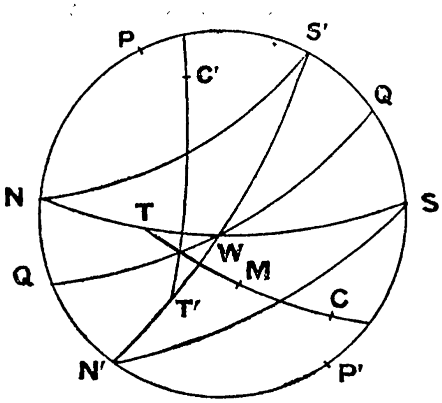

FIG. 4.

azimuth to which it is directed, and the azimuth and Z.D. of the perpendicular drawn from the centre of the cœlostat to the axis of the telescope. If the latter coordinates are A and Z respectively, the first-mentioned azimuth is $A \pm \frac{1}{2}\pi$, where the distinction between the two signs at our disposal corresponds to a complete reversal of the telescope. It is clear that the mirror (by which is meant here the subsidiary plane mirror) must move along the axis of the telescope. This axis, together with the centre of the cœlostat, determines a plane, or rather a half-plane, in which lie all the rays which can be brought into the centre of the field of the telescope. To this half-plane corresponds on the sphere (fig. 4) a great semicircle, with its extremities T, R on the horizon NWS, inclined to this circle at an angle $Z - \frac{1}{2}\pi$, and having for its middle point the point C, which represents the perpendicular from the centre of the cœlostat to the axis of the telescope. If a point on the semicircle has the N.P.D. Δ' and the hour-angle h', while the N.P.D. and hour-angle of a star

reflected in the direction of the point are Δ and h, and if the hour-angle of the normal to the cœlostat is H, then

$$\Delta' = \pi - \Delta, \qquad h' = 2\text{H} - h$$

Each point on the axis of the telescope corresponds to a definite declination, and the effect of turning the cœlostat in any manner is to bring different hour-angles into view.

The relations between the position of any point on the sphere and of its reflexion in the cœlostat are equivalent to two steps which it is convenient to consider as made successively. These consist in passing, first, to the image of the point in the plane of the equator; and, secondly, in passing to the image of this image in the plane of the meridian which contains the normal to the cœlostat. Now let N′WS′ be the image of the horizon in the equatorial plane, and let T′C′R′ be the image of the semicircle TCR. Then clearly T′C′R′ is also a semicircle with its extremities on N′WS′, and inclined to N′WS′ at the same angle as TCR to NWS. It lies wholly on one side of the image of the horizon, and its position decides the range of declination which the telescope can command. Then by suitably choosing the position of the normal to the cœlostat any point in T′C′R′ can be brought, subject to certain limitations which must be examined, to any desired hour-angle in the sky.

14. The limitations referred to are imposed by the maximum angle of incidence on the cœlostat which can in practice be allowed. This angle cannot exceed 60°, in which case the effective aperture of the cœlostat is reduced by one half, and ought, if possible, to be less than 45°. Denoting the angle of incidence by i and its limiting value by I, we have

$$\cos i = \sin\Delta \cos \tfrac{1}{2}(h'-h) \nless \cos \text{I}$$

and so we obtain for the two values of I the following limiting values of $h' \sim h$:

I = 60°.

Δ = 30°	40°	50°	60°	70°	80°	90°	100°	110°	120°
$h' \sim h$ = $0^{\text{h}}{\cdot}0$	$5^{\text{h}}{\cdot}2$	$6^{\text{h}}{\cdot}6$	$7^{\text{h}}{\cdot}3$	$7^{\text{h}}{\cdot}7$	$7^{\text{h}}{\cdot}9$	$8^{\text{h}}{\cdot}0$	$7^{\text{h}}{\cdot}9$	$7^{\text{h}}{\cdot}7$	$7^{\text{h}}{\cdot}3$

I = 45°.

Δ = 45°	50°	60°	70°	80°	90°	100°	110°	120°
$h' \sim h$ = $0^{\text{h}}{\cdot}0$	$3^{\text{h}}{\cdot}0$	$4^{\text{h}}{\cdot}7$	$5^{\text{h}}{\cdot}5$	$5^{\text{h}}{\cdot}9$	$6^{\text{h}}{\cdot}0$	$5^{\text{h}}{\cdot}9$	$5^{\text{h}}{\cdot}5$	$4^{\text{h}}{\cdot}7$

By the nature of the case the sky in the neighbourhood of the pole to a distance equal to $\frac{1}{2}\pi - \text{I}$ cannot be brought within the field of the telescope.

15. The circumstances of the reflexion at the surface of the subsidiary mirror need explanation. The point O (fig. 5) is the centre of the cœlostat and the centre of the sphere represented in fig. 4. The axis of the telescope, which is pointed in a direc-

tion parallel to OT, is CM, C being the foot of the perpendicular from O and M the position of the mirror. Let $OC = d$, $MC = x$, and $TOM = a$, so that

$$x = d \cot a$$

The normal to the mirror MV is in the plane TOC, and the angle VMC is $\frac{1}{2}a$. The axis about which the mirror must be capable of turning is normal to the plane TOC and so is fixed in direction. The angle of incidence on the mirror being $\frac{1}{2}a$, we have $a \ngtr 2I'$, where I' is the greatest angle of incidence allowed. The ray for which $a = 0$ cannot be brought into the field of the telescope; and if an upper limit is assigned to x, the distance of the mirror from C, there is a corresponding lower limit to a.

To complete the investigation of the circumstances on which the adjustment of the mirror depends, it is necessary to find the relations between a, Δ' and h' in terms of the data which express

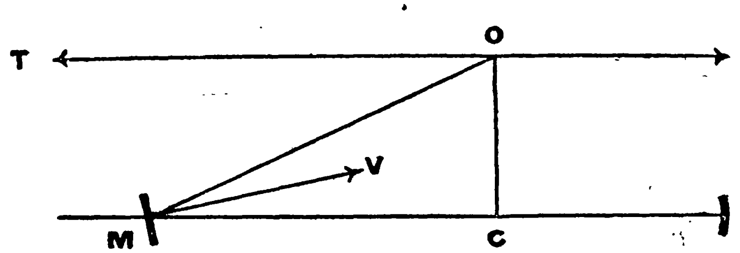

Fig. 5.

in a natural form the position of the telescope relative to the cœlostat. These are (fig. 4) $NT = B$, Z the Z.D. of C, and $PN = \phi$, the latitude. It is convenient also to introduce the dependent constants p and h_0, the N.P.D. and hour-angle of T, and τ the angle PTN, so that $PTM = \frac{1}{2}\pi + Z - \tau$. Then we obtain

$$\left.\begin{aligned} \cos \Delta' &= \cos p \cos a - \sin p \sin a \sin (Z - \tau) \\ \sin p \cot a &= -\cos p \sin (Z - \tau) + \cos (Z - \tau) \cot (h_0 - h') \end{aligned}\right\} \qquad (16)$$

where

$$\begin{aligned} \cos p &= \cos B \cos \phi \\ \cot h_0 &= -\cot B \sin \phi \\ \cot \tau &= \sin B \cot \phi \end{aligned}$$

With these, or equivalent, formulæ it is possible to tabulate in any given case the corresponding values of a, Δ' and h', and with the tables so formed to adjust the position and inclination of the mirror according to the N.P.D. required, and to make the corresponding setting of the cœlostat.

The formulæ connecting Z, A, the Z.D. and azimuth (reckoned

from N through W) of the point C with Δ_o', h_o', the N.P.D. and hour-angle of the same point, may also be useful. They are

$$\left.\begin{aligned} \cos Z &= \cos \Delta_o' \sin \phi + \sin \Delta_o' \cos \phi \cos h_o' \\ \cot A \sin h_o' &= \cot \Delta_o' \cos \phi - \sin \phi \cos h_o' \end{aligned}\right\} \quad \ldots \quad (17)$$

16. The principles discussed above enable us to examine the nature and relative advantages of different arrangements which are possible. In choosing a suitable disposition of the instruments there are three main factors to be considered: namely, (*a*) the effective range of declination within reach; (*b*) the hour-angle of the part of the sky under observation; and (*c*) the angle of incidence at the cœlostat. The two latter are, as we have seen, closely related, while the first factor depends entirely on the position of the arc T′C′ (fig. 4) in the sky. A few typical arrangements will now be briefly examined.

(1) T′ coincides with S′, and C′ is taken on S′S. From the corresponding position of TC it is clear that this means that the telescope is pointed due south, and its axis passes directly *over* the cœlostat, for C coincides with the zenith. The whole of the southern meridian is observed under the best possible conditions to a distance from the pole equal to the latitude, the limits of N.P.D. being ϕ and $\pi-\phi$. Except in very high latitudes this position is always suitable for solar work, and is in fact the one adopted for the Snow telescope.* Its maximum efficiency requires a moderately low latitude. In latitude 40°, for example, the meridian can be observed up to and beyond the zenith, and the only part of the sky which lies out of reach is the vicinity of the pole, for which the cœlostat is by its nature inadequate.

(2) T′ coincides with S′ and C′ is taken on S′P. The telescope is pointed due south and its axis passes *underneath* the cœlostat. This position supplements the former and brings under observation that part of the meridian whose limits of N.P.D. are ϕ and $\frac{1}{2}\pi - I$.

(3) Still keeping the telescope pointed south, we may place its axis on a level with and west of the cœlostat, so that C′ comes to W. For the arc S′W the limits of N.P.D. are ϕ and $\frac{1}{2}\pi$. Without making the angle of incidence at the cœlostat excessive, the available part of the sky can be observed on or near the meridian. But this case is clearly inferior to (1), and it is fairly evident that with the telescope pointed south it is impossible to place the axis so as to obtain any advantage by combining parts of the ranges in N.P.D. covered by cases (1) and (2).

(4) T′ coincides with N′ and C′ lies on N′P. The telescope is pointed due north, and its axis passes *underneath* the cœlostat. This case at first sight offers the greatest advantage with regard to the range in N.P.D., for the arc extends over N′P and corresponds to the whole of the visible sky. But if we apply the

* This is only approximately true as the telescope is at present installed on Mount Wilson. See *Astrophysical Journal*, 1905 March, p. 163.

results found in § 14 we shall find that the effective range is greatly curtailed; and even so it is necessary to have a large angle of incidence at the cœlostat, and to work at a quite impracticable hour-angle. This case presents no advantage.

(5) The telescope is pointed due west. This position may be recommended by local considerations as to available space, and these must be taken into account as well as the factors already mentioned. In this case T′ coincides with W and C′ lies on the meridian and must be placed either on QP or on QS, so that the range of sky is entirely above or below the equator. (*a*) If below, S is the best position for C′; and if this is adopted, C coincides with S′, and the perpendicular from the cœlostat to the axis of the mirror is inclined southward at an angle $2\phi-\frac{1}{2}\pi$ to the upward vertical, the range in N.P.D. being from $\frac{1}{2}\pi$ to $\pi-\phi$. (*b*) If, as would naturally be the case in a moderately high latitude, north declinations are required, C′ may be taken at a distance $\frac{1}{2}\pi-\mathrm{I}$ from P, so as to obtain the greatest range of declination possible at the most favourable hour-angle. The perpendicular from the cœlostat to the axis of the mirror is inclined southward at an angle $\phi+\mathrm{I}$ to the upward vertical. The range covered in N.P.D. extends from $\frac{1}{2}\pi-\mathrm{I}$ to $\frac{1}{2}\pi$. This case is therefore superior to (3) at places where the co-latitude is less than the greatest angle of incidence at the cœlostat which is considered allowable.

17. Enough has been said with regard to the use of a cœlostat in conjunction with a fixed horizontal telescope and a second mirror. Positions of the telescope other than horizontal will not be discussed here. But mention may be made of the case in which the telescope is directed to the zenith. Here the second mirror must be moved vertically; and if the cœlostat is placed to the north of the axis of the telescope, a considerable part of the southern meridian, interrupted at the equator itself, can be brought within the field, ranging, in latitude 45°, practically from the horizon to the zenith. The idea of a concave mirror placed a hundred feet or so beneath the surface of the earth is a strange one; but the plan might possibly avoid the defects of the horizontal position: namely, the bad definition due to the unsteadiness of the atmosphere near the ground and the variability in the focal length owing to changes in the temperature of the mirror.

18. In addition to the cœlostat we have as a particular case of the siderostat the polar heliostat of Fahrenheit, which may be regarded as the prototype of the modern coudé form of equatorial. The essential difference between the combination of a telescope and a polar heliostat and the Sheepshanks telescope at Cambridge, for example, consists merely in the order of the mirror and the object-glass. The inconvenience of the instrument arises from the rotation of the field which is obviated by the cœlostat. It is also necessary to place the axis of the telescope parallel to the polar axis. This position can, however, be

avoided by the use of a second mirror fixed so as to deflect the rays from the heliostat in any desired direction. In this way a polar heliostat can be used in conjunction with either a horizontal or a vertical telescope.

19. The contents of this paper may be summarised thus:

(1) §§ 1 and 2 are introductory.

(2) §§ 3 and 4 deal with the geometry of a mirror caused to rotate with uniform angular velocity about the polar axis, which is inclined at a constant angle to the plane of the mirror.

(3) §§ 5–8 contain a discussion of the properties of a mirror similarly mounted, but rotated with such a (variable) velocity that the image of a particular star is maintained on the instrumental meridian.

(4) §§ 9–11 deal with the motion of a siderostat considered as a mirror capable of rotation about any two orthogonal axes, one of which is fixed in the plane of the instrumental meridian.

(5) § 12 describes a linkage which is capable of producing the motions required by a siderostat which is mounted equatorially.

(6) §§ 13–15 deal with the geometrical conditions on which the efficient use of a second mirror in conjunction with a cœlostat depends when the telescope is fixed in a horizontal position, and with the necessary adjustment in the position of the mirror.

(7) § 16 is devoted to an examination of certain typical positions of the telescope.

(8) §§ 17 and 18 refer to the use of a cœlostat with a telescope pointed to the zenith and to the possible advantage of a polar heliostat combined with a fixed mirror.

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The Optical Sine-condition. By A. E. Conrady.

The remarkable theorem which forms the subject of this paper was brought to the notice of astronomers in general by the stipulations agreed upon at the Paris Congress for standardising the construction of the astrographic telescopes; for, on the recommendation of Dr. Steinheil, one of the German delegates, it was laid down that the objectives must fulfil the sine-condition, so as to insure the formation of symmetrical images in the outer part of the relatively large field that had to be covered.

A short history of this theorem and a simple proof of it may therefore be acceptable.

Owing to its close association with the complicated defect known to opticians as "coma," it is a matter of course that this